ON THE OSCILLATIONS OF A BODY ABOVE THE INTERFACE BETWEEN TWO LIQUIDS

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Kochin [1] solved the plane linear problem of waves on the free surface of a heavy liquid of infinite depth, caused by vibrations of a body immersed in the liquid. Khaskind [2] used Kochin's method [1] to study a similar problem for a liquid of finite depth. In [3] Kochin's method was used for study of the case when there is another layer of lighter liquid of finite depth with a free surface, overlying the surface of a liquid of infinite depth in which the body is oscillating. Below we consider the analogous problem for oscillations of a body in the upper layer of liquid.

1. Formulation of the problem. Let the contour of the body Γ of arbitrary shape perform periodic oscillations in a layer of homogeneous incompressible liquid of density ρ_1 and finite thickness d, which lies on another liquid of density $\rho_2 (\rho_2 > \rho_1)$ and infinite depth (Fig. 1). To the usual assumptions of the linear theory of waves we add the requirement that the waves diverge horizontally from the body on both sides. By virtue of the linearity of the problem we consider only harmonic oscillations of the contour Γ with the specified frequency k

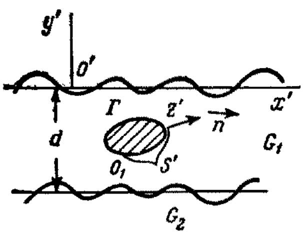

Fig. 1.

$$v_n'(s', t') = u_1'(s') \cos kt' + u_2'(s') \sin kt' \qquad (1.1)$$

where v_n' is the normal component of the velocity of an arbitrary point of the contour Γ, which corresponds to the arc length s', measured from a certain fixed point O_1 on Γ. Then assuming that the oscillations of the liquid are steady and potential, we have the complex velocity potential in the form ($z' = x' + iy'$)

$$W_j'(z', t') = w_{j1}'(z')\cos kt' + w_{j2}'(z')\sin kt' \tag{1.2}$$

Here and in what follows $j = 1$ and $j = 2$ for the upper and lower liquid, respectively.

Transforming to dimensionless quantities according to the formulas

$$z' = zd, \quad t' = \frac{t}{k}, \quad w_{jm}' = w_{jm}kd^2, \quad \frac{\rho_2}{\rho_1} = \rho_0, \quad \frac{k^2 d}{g} = \nu \tag{1.3}$$

we obtain the following problem for the functions $\bar{v}_{jm}(z) = dw_{jm}(z)/dz$.

To find the functions $\bar{v}_{1m}(z)$ and $\bar{v}_{2m}(z)$, analytic in the respective regions G_1 and G_2 (G_1 is the region in the strip $-1 < \operatorname{Im} z < 0$ outside the contour Γ, G_2 is the half-plane $\operatorname{Im} z < -1$) and satisfying the conditions:

a) $\operatorname{Im}\left[\dfrac{d\bar{v}_{1m}}{dz} + i\nu\bar{v}_{1m}\right] = 0$ when $y = 0$

b) $\operatorname{Im}[\bar{v}_{1m} - \bar{v}_{2m}] = 0$ when $y = -1$

c) $\operatorname{Im}\left[\left(\dfrac{d\bar{v}_{1m}}{dz} + i\nu\bar{v}_{1m}\right) - \rho_0\left(\dfrac{d\bar{v}_{2m}}{dz} + i\nu\bar{v}_{2m}\right)\right] = 0$ when $y = -1$

d) The functions $\bar{v}_{jm}(z)$ are bounded in G_j and $\bar{v}_{2m}(z) \to 0$ when $y \to -\infty$.

e) The waves on the free boundary and on the interface diverge on both sides from the body profile Γ.

f) $\operatorname{Re}[\bar{v}_{1m}e^{i\theta}] = u_m(s)$ on Γ

Here θ is the angle between the exterior normal **n** to the contour Γ and the x-axis.

On the basis of (1.2) and (1.3) we have for the complex velocities of the liquids

$$\bar{v}_j(z, t) = dW_j/dz = \bar{v}_{j1}(z)\cos t + \bar{v}_{j2}(z)\sin t \tag{1.4}$$

We find the free boundary and the interface according to the formulas

$$\begin{aligned}\delta_1(x, t) &= -\operatorname{Im}[\bar{v}_{11}(z)\cos t - \bar{v}_{12}(z)\sin t]_{y=0}\\ \delta_2(x, t) &= -\operatorname{Im}]\bar{v}_{21}(z)\cos t - \bar{v}_{22}(z)\sin t]_{y=1}\end{aligned} \tag{1.5}$$

2. Construction of the solution. 1. Let us represent the required functions $\bar{v}_{jm}(z)$ in the form

$$\bar{v}_{jm}(z) = \Omega_{jm}(z) + F_{jm}(z) \tag{2.1}$$

where $\Omega_{jm}(z)$ are functions, analytic in their regions G_j and satisfying conditions (a) to (d), whilst the functions $F_{1m}(z)$ and $F_{2m}(z)$ are analytic respectively in the regions G_1' and G_2 (G_1' is the strip $-1 < \operatorname{Im} z < 0$) and also satisfy conditions (a) to (d). Conditions (e) and (f) are to be satisfied, not by the functions $\Omega_{jm}(z)$ and $F_{jm}(z)$ separately, but by their sums $\bar{v}_{jm}(z)$ from equation (2.1).

2. Let us find the functions $\omega_{jm}(z)$ with the following properties. The functions $\omega_{1m}(z)$ are analytic in the region G_1' everywhere except at the point $\zeta = \xi + i\eta$, where they have a polar singularity with the residue $N_m = N_m' + iN_m''$. The functions $\omega_{2m}(z)$ are analytic in G_2. The functions $\omega_{1m}(z)$ and $\omega_{2m}(z)$ satisfy conditions (a) to (d).

Let us isolate the singularity at the point ζ, setting

$$\omega_{1m}(z) = \tau_{1m}(z) + \varepsilon_m(z), \qquad \varepsilon_m(z) = \frac{N_m}{z-\zeta} + \frac{\overline{N}_m}{z-\bar{\zeta}} \tag{2.2}$$

Then the functions $\tau_{1m}(z)$ and $\omega_{2m}(z)$ are analytic in the regions G_1' and G_2 respectively, whilst their real parts $\mu_{jm}(x, y)$ satisfy conditions

$$\frac{\partial\mu_{1m}}{\partial y} - \nu\mu_{1m} = f_{1m}(x) \quad \text{when } y = 0, \qquad \frac{\partial\mu_{1m}}{\partial y} - \frac{\partial\mu_{2m}}{\partial y} = f_{2m}(x) \quad \text{when } y = -1$$
$$\left(\frac{\partial\mu_{1m}}{\partial y} - \nu\mu_{1m}\right) - \rho_0\left(\frac{\partial\mu_{2m}}{\partial y} - \nu\mu_{2m}\right) = f_{3m}(x) \quad \text{when } y = -1 \tag{2.3}$$

and the functions μ_{1m} and μ_{2m} are bounded in their regions G_1' and G_2 respectively, and $\mu_{2m} \to 0$ when $y \to -\infty$ (condition (d)).

Here

$$f_{2m}(x) = \operatorname{Im}\frac{d\varepsilon_m}{dz} \quad \text{when } y = -1, \qquad f_{lm}(x) = \operatorname{Im}\left[\frac{d\varepsilon_m}{dz} + i\nu\varepsilon_m\right]$$

when $y = 0$ for $l = 1$, when $y = -1$ for $l = 3$.

Representing the functions $f_{lm}(x)$ by Fourier integrals, seeking the functions $\tau_{1m}(z)$ and $\omega_{2m}(z)$ in the form of Fourier integrals, and making use of condition (2.3), we obtain

$$\omega_{1m}(z) = \frac{N_m}{z-\zeta} + \frac{\overline{N}_m}{z-\bar{\zeta}} + i\int_0^\infty [2iN_m A \sin\lambda(z-\zeta) + \overline{N}_m Be^{-i\lambda(z-\bar{\zeta})} +$$
$$+ \overline{N}_m Ce^{i\lambda(z-\bar{\zeta})}]\,d\lambda, \quad \omega_{2m}(z) = i\int_0^\infty [N_m De^{-i\lambda(z-\zeta)} + \overline{N}_m Ee^{-i\lambda(z-\bar{\zeta})}]\,d\lambda \tag{2.4}$$

Here

$$A = \frac{r(\lambda+\nu)}{T(\lambda)}e^{-2\lambda}, \quad B = \frac{4\nu^2 - r(\lambda-\nu)\,[2\nu + (\lambda+\nu)\,e^{-2\lambda}]}{(\lambda-\nu)\,T(\lambda)} \tag{2.5}$$

$$C = \frac{r(\lambda-\nu)}{T(\lambda)}\,e^{-2\lambda}, \qquad D = \frac{2\nu}{T(\lambda)}, \qquad E = \frac{2\nu(\lambda+\nu)}{(\lambda-\nu)\,T(\lambda)}$$

$$T(\lambda) = 2\nu + r\,[(\lambda+\nu)\,e^{-2\lambda} - \lambda + \nu)], \qquad r = \rho_0 - 1$$

For any $\nu > 0$ equation $T(\lambda) = 0$ has one root $\lambda = \lambda_0 > 0$. Consequently for the integrals in expressions (2.4) it is appropriate to take their principal values in the Cauchy sense.

3. Let us distribute along the contour Γ polar singularities with residues of $N_m = \gamma_m(\sigma)\,d\sigma/2\pi$, where σ is the arc length of the contour Γ corresponding to the point $\zeta(\sigma)$ of this contour.

Then, integrating expressions (2.4) along the contour Γ, we obtain the functions

$$\Omega_{1m}(z) = \frac{1}{2\pi}\int_\Gamma \gamma_m(\sigma)\left\{\frac{1}{z-\zeta} + \frac{1}{z-\bar\zeta} + i\int_0^\infty [2iA\sin\lambda(z-\zeta) + Be^{-i\lambda(z-\bar\zeta)} + \right.$$
$$\left. + Ce^{i\lambda(z-\bar\zeta)}]\,d\lambda\right\}d\sigma, \qquad \Omega_{2m}(z) = \frac{i}{2\pi}\int_\Gamma \gamma_m(\sigma)\left\{\int_0^\infty [De^{-i\lambda(z-\zeta)} + Ee^{-i\lambda(z-\bar\zeta)}]\,d\lambda\right\}d\sigma \tag{2.6}$$

which, by virtue of the linearity of the problem, are analytic in the regions G_1 and G_2, respectively, and satisfy conditions (a) to (d).

4. Starting from the form of a particular solution of Laplace's equation and from conditions (a) to (d), we find the following functions, analytic in the regions $G_1{}'$ and G_2:

$$F_{1m}(z) = A_m e^{-i\lambda_0 z} + B_m e^{-i\nu z} + C_m e^{i\lambda_0 z}, \qquad F_{2m}(z) = D_m e^{-i\lambda_0 z} + E_m e^{-i\nu z} \tag{2.7}$$

where $A_m, \ldots, E_m$ are certain constants. Then, from (2.1) we obtain

$$\bar v_{1m}(z) = \frac{1}{2\pi}\int_\Gamma \gamma_m(\sigma)\left\{\frac{1}{z-\zeta} + \frac{1}{z-\bar\zeta} + i\int_0^\infty [2iA\sin\lambda(z-\zeta) + Be^{-i\lambda(z-\bar\zeta)} + \right.$$
$$\left. + Ce^{i\lambda(z-\bar\zeta)}]\,d\lambda\right\}d\sigma + A_m e^{-i\lambda_0 z} + B_m e^{-i\nu z} + C_m e^{i\lambda_0 z} \tag{2.8}$$

$$\bar v_{2m}(z) = \frac{i}{2\pi}\int_\Gamma \gamma_m(\sigma)\left\{\int_0^\infty [De^{-i\lambda(z-\zeta)} + Ee^{-i\lambda(z-\bar\zeta)}]\,d\lambda\right\}d\sigma + D_m{}^{-i\lambda_0 z} + E_m e^{-i\nu z}$$

From the linearity of the problem it follows that the functions

$\bar{v}_{jm}(z)$ from (2.8) are analytic in G_j and satisfy conditions (a) to (d). The functions $\gamma_m(\sigma)$ and the constants A_m, ..., E_m we shall find from conditions (e) and (f).

3. Determination of the coefficients A_m, ..., E_m. We shall make use of condition (e) for finding the quantities A_m, ..., E_m. In expressions (2.8) let us pass from integrals in the sense of the principal value to integrals along the contours L_-, L_+, L_-' and L_+' in the plane of the complex variable λ (Figs. 2 and 3).

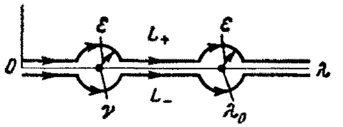

Fig. 2.

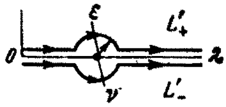

Fig. 3.

Then, writing

$$i \int_{\Gamma} \gamma_m(\sigma)\, e^{-i\lambda\zeta}\, d\sigma = H_m(\lambda) \tag{3.1}$$

we obtain

$$\bar{v}_{1m}(z) = \int_{\Gamma} \gamma_m(\sigma) \left\{ \frac{1}{2\pi} \left[\frac{1}{z-\zeta} + \frac{1}{z-\bar{\zeta}} \right] + \frac{i}{2\pi} \left[\int_{L_{\mp}'} (Ae^{i\lambda(z-\zeta)} + Ce^{i\lambda(z-\bar{\zeta})})\, d\lambda - \right.\right.$$

$$\left.\left. - \int_{L_{\pm}'} Ae^{-i\lambda(z-\zeta)}\, d\lambda + \int_{L_{\pm}} Be^{-i\lambda(z-\bar{\zeta})}\, d\lambda \right] \right\} d\sigma + R_m^{\mp}(z) \tag{3.2}$$

$$R_m^{\mp}(z) = \{A_m \mp i\,[A_0 H_m(-\lambda_0) + B_0 \overline{H_m(\lambda_0)}]\}\, e^{-i\lambda_0 z} +$$

$$+ \{C_m \mp i\,[A_0 H_m(\lambda_0) - C_0 \overline{H_m(-\lambda_0)}]\}\, e^{i\lambda_0 z} + \{B_m \mp iB_\nu \overline{H_m(\nu)}\}\, e^{-i\nu z} \tag{3.3}$$

$$A_0 = \frac{r(\lambda_0+\nu)}{2T'(\lambda_0)}\, e^{-2\lambda_0}, \qquad B_0 = -\frac{r(\lambda_0+\nu)^2}{2(\lambda_0-\nu)\,T'(\lambda_0)}\, e^{-2\lambda_0} \tag{3.4}$$

$$B_\nu = \frac{\nu}{1+re^{-2\nu}}, \qquad C_0 = \frac{r(\lambda_0-\nu)}{2T'(\lambda_0)}\, e^{-2\lambda_0}, \qquad T'(\lambda_0) = \left(\frac{dT}{d\lambda}\right)_{\lambda=\lambda_0}$$

Here the upper sign is taken when $x < 0$, and the lower sign when $x > 0$.

It is easy to verify that in formulas (3.2)

$$\bar{v}_{1m}(z) \to R_m^{\mp}(z) \quad \text{when } |x| \to \infty$$

According to formulas (1.4) we obtain asymptotic values for the total velocity

$$\bar{v}_1(z, t) \approx R_1^{\mp}(z)\cos t + R_2^{\mp}(z)\sin t \quad \text{when } x \to \mp\infty$$

Consequently, condition (1.5) for the upper liquid can be represented in the form

$$\bar{v}_1(z, t) \approx A_{\mp} e^{-i\lambda_0 z \mp it} + B_{\mp} e^{-i\nu z \mp it} + C_{\mp} e^{i\lambda_0 z \pm it} \quad \text{when } x \to \mp\infty \tag{3.5}$$

From this condition we obtain

$$A_1 = A_0 H_2(-\lambda_0) + B_0 \overline{H_2(\lambda_0)}, \qquad A_2 = -A_0 H_1(-\lambda_0) - B_0 \overline{H_1(\lambda_0)}$$
$$B_1 = B_\nu \overline{H_2(\nu)}, \quad B_2 = -B_\nu \overline{H_1(\nu)}, \tag{3.6}$$
$$C_1 = -A_0 H_2(\lambda_0) + C_0 \overline{H_2(-\lambda_0)}, \qquad C_2 = A_0 H_1(\lambda_0) - C_0 \overline{H_1(-\lambda_0)}$$

$$A_- = -iA_0 h_1(-\lambda_0) - iB_0 \overline{h_2(\lambda_0)}, \quad A_+ = iA_0 h_2(-\lambda_0) + iB_0 \overline{h_1(\lambda_0)}$$
$$B_- = -iB_\nu \overline{h_2(\nu)}, \quad B_+ = iB_\nu \overline{h_1(\nu)}, \tag{3.7}$$
$$C_- = -iA_0 h_2(\lambda_0) + iC_0 \overline{h_1(-\lambda_0)}, \qquad C_+ = iA_0 h_1(\lambda_0) - iC_0 \overline{h_2(-\lambda_0)}$$

Here

$$h_1(\lambda) = H_1(\lambda) + iH_2(\lambda), \qquad h_2(\lambda) = H_1(\lambda) - iH_2(\lambda) \tag{3.8}$$

For the lower liquid we find in a completely analogous manner that

$$\bar{v}_2(z, t) \approx D_{\mp} e^{-i\lambda_0 z \mp it} + E_{\mp} e^{-i\nu z \mp it} \quad \text{when } x \to \mp\infty \tag{3.9}$$

$$D_1 = -D_0 H_2(-\lambda_0) + E_0 \overline{H_2(\lambda_0)} \qquad D_2 = D_0 H_1(-\lambda_0) - E_0 \overline{H_1(\lambda_0)}$$
$$E_1 = B_\nu \overline{H_2(\nu)}, \qquad E_2 = -B_\nu \overline{H_1(\nu)}, \qquad D_- = iD_0 h_1(-\lambda_0) - iE_0 \overline{h_2(\lambda_0)} \tag{3.10}$$
$$D_+ = -iD_0 h_2(-\lambda_0) + iE_0 \overline{h_1(\lambda_0)}, \quad E_- = -iB_\nu \overline{h_2(\nu)}, \quad E_+ = iB_\nu \overline{h_1(\nu)}$$

Here

$$D_0 = \frac{\nu}{T'(\lambda_0)}, \qquad E_0 = \frac{\nu(\lambda_0 + \nu)}{(\lambda_0 - \nu)T'(\lambda_0)} \tag{3.11}$$

We shall now explain the meaning of the functions $H_m(\lambda)$ from (3.1). We can prove the equality

$$\int_{\Gamma_1} \bar{v}_{1m}(z) e^{-i\lambda z} dz = i \int_{\Gamma} \gamma_m(\sigma) e^{-i\lambda\zeta} d\sigma$$

or

$$H_m(\lambda) = \int_{\Gamma_1} \bar{v}_{1m}(z) e^{-i\lambda z} dz \tag{3.12}$$

where Γ_1 is any closed contour in the upper liquid including inside itself

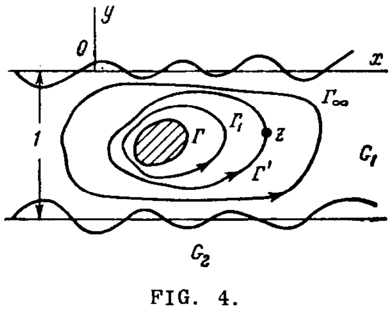

FIG. 4.

the contour Γ (Fig. 4).

For this it is sufficient to substitute the values of the functions $\bar{v}_{1m}(z)$ from (2.8) and make use of the known properties of analytic functions and equation (3.1). Consequently, the functions $H_m(\lambda)$ are generalised circulations for certain fictitious fluid flows with the complex potentials $w_{jm}(z)$, i.e. they are the functions of Kochin [1].

4. Determination of the functions $\gamma_m(\sigma)$. 1. We find the functions $\gamma_m(\sigma)$ with the help of condition (f). For definiteness, we choose the origin of coordinates 0 so that Re $z > 0$ for points z on the contour Γ (Figs. 1 and 4). Then, using (3.2), (3.3), (3.6) and (3.7), we obtain

$$\bar{v}_{1m}(z) = \int_\Gamma \gamma_m(\sigma)\left\{\frac{1}{2\pi}\left[\frac{1}{z-\zeta}+\frac{1}{z-\bar{\zeta}}\right]+\frac{i}{2\pi}\left[\int_{L_+'}(Ae^{i\lambda(z-\zeta)}+Ce^{i\lambda(z-\bar{\zeta})})\,d\lambda - \right.\right.$$
$$\left.\left. - \int_{L_-'} Ae^{-i\lambda(z-\zeta)}\,d\lambda + \int_{L_-} Be^{-i\lambda(z-\bar{\zeta})}\,d\lambda\right]\right\}d\sigma + A_m'e^{i\lambda_0 z} + B_m'e^{-i\nu z} + C_m'e^{i\lambda_0 z} \tag{4.1}$$

Here

$$A_1' = A_+, \quad B_1' = B_+, \quad C_1' = C_+, \quad A_2' = iA_1', \quad B_2' = iB_1', \quad C_2' = -iC_1' \tag{4.2}$$

Let the contour Γ be simple and have continuous curvature. Substituting the functions $\bar{v}_{1m}(z)$ from (4.1) in condition (f) and using the properties of integrals of Cauchy type, we obtain the integral equations

$$\gamma_m(s) = -K\gamma_m + f_m(s) \tag{4.3}$$

where the arc length s corresponds to the point z on the contour Γ and

$$K\gamma_m \equiv \int_\Gamma K(s,\sigma)\,\gamma_m(\sigma)\,d\sigma, \quad K(s,\sigma) = \frac{1}{\pi}\,\mathrm{Re}\left\{ie^{i\theta}\left[\int_{L_+'}(Ae^{i\lambda(z-\zeta)}+Ce^{i\lambda(z-\bar{\zeta})})\,d\lambda - \right.\right.$$
$$\left.\left. - \int_{L_-'} Ae^{-i\lambda(z-\zeta)}d\lambda + \int_{L_-} Be^{-i\lambda(z-\bar{\zeta})}d\lambda\right] + \frac{e^{i\theta}}{z-\zeta} + \frac{e^{i\theta}}{z-\bar{\zeta}}\right\} \tag{4.4}$$

$$f_m(s) = 2u_m(s) - 2\mathrm{Re}\,[e^{i\theta}(A_m'e^{-i\lambda_0 z} + B_m'e^{-i\nu z} + C_m'e^{i\lambda_0 z})] \tag{4.5}$$

It is evident that we obtain linear integral equations of Fredholm type with a continuous kernel.

If the contour Γ oscillates but does not deform, then

$$\int_\Gamma \boldsymbol{u}_m(s)\,ds = 0, \qquad \int_\Gamma K(s,\sigma)\,ds = 1$$

Then, integrating both parts of equation (4.3) with respect to $s \in [0, b]$, where b is the length of the contour Γ, we obtain

$$\int\limits_{\Gamma} \gamma_m(s)\,ds = 0 \tag{4.6}$$

On the basis of (4.6), equations (4.3) are equivalent to equations

$$\gamma_m(s) = -K_0\gamma_m + f_m(s) \qquad (K_0 = K - 1/b) \tag{4.7}$$

The solutions of these equations can be sought by the method of successive approximations

$$\gamma_m(s) = \sum_{k=0}^{\infty}(-1)^k q_{mk}(s), \qquad q_{mk}(s) = K_0 q_{m,\,k-1}, \qquad q_{m0}(s) = f_m(s) \tag{4.8}$$

With the help of the principle of compressive transformations [4] we establish the condition for convergence of the process (4.8) to a unique solution. We shall seek the solution of equation (4.7) in the class M of functions, bounded when $s \in [0,\ b]$. It is easy to prove that if $\gamma_m \in M$, then also $F\gamma_m \in M$, where $F\gamma_m = -K_0\gamma_m + f_m(s)$. Moreover, if $\gamma_{m1} \in M$ and $\gamma_{m2} \in M$, then

$$|F\gamma_{m1} - F\gamma_{m2}| \leqslant \|\gamma_{m1} - \gamma_{m2}\|_M \int\limits_{\Gamma} |K_0(s,\sigma)|\,d\sigma \quad (\|\gamma_m\|_M = \sup|\gamma_m(s)| \text{ when } s \in [0,\ b])$$

This means that if

$$\int\limits_{\Gamma} |K_0(s,\sigma)|\,d\sigma \leqslant p < 1 \tag{4.9}$$

then the operator F is an operator of approximation.

Accordingly, for contours Γ satisfying condition (4.9) there exist solutions, and moreover unique solutions, of equations (4.7), and hence also of equations (4.3). From (4.4) we have

$$\begin{aligned}\int\limits_{\Gamma}\Big|K_0(s,\sigma)\Big|\,d\sigma \leqslant \frac{1}{\pi}\int\limits_{\Gamma}\left|\operatorname{Re}\left(\frac{e^{i\theta}}{z-\zeta}\right) - \frac{\pi}{b}\right|d\sigma + \frac{b}{\pi}\Big\{\sup_{s,\,\sigma}\Big|\int\limits_{L_+'}[Ae^{i\lambda(z-\zeta)} + \\ + Ce^{i\lambda(z-\bar\zeta)}]\,d\lambda - \int\limits_{L_-'} Ae^{-i\lambda(z-\zeta)}\,d\lambda + \int\limits_{L_-} Be^{-i\lambda(z-\bar\zeta)}\,d\lambda\Big| + \frac{1}{2a}\Big\} \quad (s,\sigma \in [0,\ b])\end{aligned} \tag{4.10}$$

where a is the least distance from the x-axis to points on the contour Γ. If the contour Γ is a circle, then

$$\operatorname{Re}\frac{e^{i\theta}}{z-\zeta} - \frac{\pi}{b} = 0$$

Accordingly, from the inequality (4.10) we see that condition (4.9) is automatically fulfilled for contours Γ for which the length b is sufficiently small in comparison with unity and which have a shape sufficiently close to a circle with radius $b/2\pi$.

2. Let us find the constants A_m', B_m' and C_m', appearing in expression (4.5) for the functions $f_m(s)$ and appearing, consequently, in solution (4.8). Let us represent the functions $\gamma_m(s)$ in the form

$$\gamma_m(s) = \gamma_m^\circ(s) + \mathrm{Re}\,[\gamma(s, \lambda_0) A_m' + \gamma(s, \nu) B_m' + \gamma(s, -\lambda_0) C_m'] \qquad (4.11)$$

where the functions $\gamma_m^\circ(s)$ and $\gamma(s, \lambda)$ are solutions of equations

$$\gamma_m^\circ(s) = -K\gamma_m^\circ + 2u_m(s), \qquad \gamma(s, \lambda) = -K\gamma - 2e^{i\theta - i\lambda z} \qquad (4.12)$$

and λ takes the values λ_0, ν and $-\lambda_0$. Evidently, the solutions of equations (4.12) exist also under condition (4.9). Substituting $\gamma_m(s)$ from (4.11) in formulas (3.8), we obtain

$$\begin{aligned} h_1(\lambda) &= H_1^\circ(\lambda) + iH_2^\circ(\lambda) + h(\lambda, \lambda_0)\overline{A_1}' + h(\lambda, \nu)\overline{B_1}' + H(\lambda, -\lambda_0) C_1' \\ h_2(\lambda) &= H_1^\circ(\lambda) - iH_2^\circ(\lambda) + H(\lambda, \lambda_0) A_1' + H(\lambda, \nu) B_1' + h(\lambda, -\lambda_0)\overline{C_1}' \end{aligned} \qquad (4.13)$$

Here

$$H_m^\circ(\lambda) = i\int_\Gamma \gamma_m^\circ(\sigma) e^{-i\lambda\zeta} d\sigma, \qquad h(\lambda, \Lambda) = i\int_\Gamma \overline{\gamma(\sigma, \Lambda)} e^{-i\lambda\zeta} d\sigma$$

$$H(\lambda, \Lambda) = i\int_\Gamma \gamma(\sigma, \Lambda) e^{-i\lambda\zeta} d\sigma \qquad (4.14)$$

whilst Λ takes the values λ_0, ν and $-\lambda_0$. Substituting $h_1(\lambda)$ and $h_2(\lambda)$ from (4.13) in formulas (3.7) and taking account of (4.2), we find that

$$A_1' = \Delta_A / \Delta, \qquad B_1' = \Delta_B / \Delta, \qquad C_1' = \overline{\Delta}_C / \overline{\Delta} \qquad (4.15)$$

where

$$\begin{aligned} \Delta_A &= p_1(b_2c_3 - b_3c_2) + p_2(b_3c_1 - b_1c_3) + p_3(b_1c_2 - b_2c_1) \\ \Delta_B &= p_1(a_3c_2 - a_2c_3) + p_2(a_1c_3 - a_3c_1) + p_3(a_2c_1 - a_1c_2) \\ \Delta_C &= p_1(a_2b_3 - a_3b_2) + p_2(a_3b_1 - a_1b_3) + p_3(a_1b_2 - a_2b_1) \\ \Delta &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \end{aligned} \qquad (4.16)$$

$$\begin{gathered} a_1 = i\,[A_0H(-\lambda_0, \lambda) + B_0\overline{h(\lambda_0, \lambda_0)}] - 1, \qquad a_3 = i\,[D_0H(-\lambda_0, \lambda_0) - A_0\overline{h(\lambda_0, \lambda_0)}] \\ a_2 = iB_\nu\overline{h(\nu, \lambda_0)}, \qquad b_1 = i\,[A_0H(-\lambda_0, \nu) + B_0\overline{h(\lambda_0, \nu)}], \qquad b_2 = iB_\nu\overline{h(\nu, \nu)} - 1 \\ b_3 = i\,[D_0H(-\lambda_0, \nu) - A_0\overline{h(\lambda_0, \nu)}], \qquad c_1 = i\,[A_0h(-\lambda_0, -\lambda_0) + B_0\overline{H(\lambda_0, -\lambda_0)}] \\ c_2 = iB_\nu\overline{H(\nu, -\lambda_0)}, \qquad c_3 = i\,[D_0h(-\lambda_0, -\lambda_0) - A_0\overline{H(\lambda_0, -\lambda_0)}] - 1 \qquad (4.17) \\ p_1 = -i\,[A_0h_0 + B_0\overline{h(\lambda_0)}], \qquad p_2 = -iB_\nu\overline{h(\nu)}, \qquad p_3 = i\,[A_0\overline{h(\nu)} - D_0h_0] \end{gathered}$$

Here

$$h_0 = H_1^\circ(-\lambda_0) - iH_2^\circ(-\lambda_0), \qquad h(\lambda) = H_1^\circ(\lambda) + iH_2^\circ(\lambda)$$

and λ takes the values λ_0 and ν.

5. Derivation of the fundamental formulas of the problem. 1. Comparing equations (3.5) and (3.9) with equations (1.4), we can find the asymptotic values of the functions $\bar{v}_{jm}(z)$ when $x \to \mp\infty$. Substituting the latter in formulas (1.5), we obtain the asymptotic expressions for the wave profiles of the free boundary and the interface, respectively

$$\delta_1(x, t) \approx \mathrm{Im}\,\{\mp i\,[(A_\mp - \overline{C}_\mp)\, e^{-i\lambda_0 x \mp it} + B_\mp e^{-i\nu x \mp it}]\} \quad (x \to \mp\infty) \qquad (5.1)$$
$$\delta_2(x, t) \approx \mathrm{Im}\,\{\mp i\,[e^{-\lambda_0} D_\mp e^{-i\lambda_0 x \mp it} + e^{-\nu} E_\mp e^{-i\nu x \mp it}]\} \qquad (x \to \mp\infty)$$

Accordingly, on the free boundary and on the interface far from the contour of the body Γ there diverge on both sides of it waves of two forms with wave lengths $2\pi/\lambda_0$ and $2\pi/\nu$ and with amplitudes $\alpha_j{\mp}$ and $\beta_j{\mp}$ respectively, where

$$\alpha_{1\mp} = |A_\mp - \overline{C}_\mp|, \qquad \beta_{1\mp} = |B_\mp|, \qquad \alpha_{2\mp} = |D_\mp|\, e^{-\lambda_0}, \qquad \beta_{2\mp} = |E_\mp|\, e^{-\nu} \qquad (5.2)$$

Equating amplitudes, we obtain relations

$$\alpha_{1\mp} = re^{-\lambda_0}\alpha_{2\mp}, \qquad \beta_{2\mp} = e^{-\nu}\beta_{1\mp} \qquad (5.3)$$

Hence it follows that the waves with length $2\pi/\lambda_0$ develop, basically, on the interface, whilst the waves with length $2\pi/\nu$ develop on the free surface.

2. The mean values $\langle M \rangle$, $\langle X \rangle$, $\langle Y \rangle$ over a period of oscillation for the principal moment M and for the projections X and Y of the resultant vector of the forces of the liquid pressure on the contour of the body Γ will be sought according to Kochin's formulas [1]

$$\langle Y \rangle + i\,\langle X \rangle = -\frac{1}{4}\int_{\Gamma'} \{[\bar{v}_{11}(z)]^2 + [\bar{v}_{12}(z)]^2\}\, dz$$
$$\langle M \rangle = -\frac{1}{4}\,\mathrm{Re}\left(\int_{\Gamma'} \{[\bar{v}_{11}(z)]^2 + [\bar{v}_{12}(z)]^2\}\, z\, dz\right) \qquad (5.4)$$

where Γ' is any contour lying in the region G_1 and enclosing the contour Γ.

Let us transform formulas (5.5). To do this, let us introduce in region G_1 contours Γ_1 and Γ_∞ as shown in Fig. 4. Then for the functions $\bar{v}_{1m}(z)$, analytic and single-values in the region G_1, we find from Cauchy's formula

$$\bar{v}_{1m}(z) = U_m(z) + V_m(z) \qquad (5.5)$$

where the point z lies on the contour Γ' and

$$U_m(z) = \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{\bar{v}_{1m}(\zeta)\,d\zeta}{z-\zeta}, \qquad V_m(z) = -\frac{1}{2\pi i}\int\limits_{\Gamma_\infty} \frac{\bar{v}_{1m}(\zeta)\,d\zeta}{z-\zeta} \tag{5.6}$$

It is obvious that the functions $U_m(z)$ are analytic outside the contour Γ_1 and are of order z^{-1} when $z \to \infty$, whilst the functions $V_m(z)$ are analytic everywhere inside the contour Γ_∞. Moreover, Γ_1 can be drawn arbitrarily close to the contour of the body Γ, whilst Γ_∞ can be drawn arbitrarily close to the straight lines $y = 0$ and $y = -1$.

Making use of the properties of analytic functions, we obtain

$$\langle Y\rangle + i\langle X\rangle = -\frac{1}{2}\int\limits_{\Gamma'} [\bar{v}_{11}(z)V_1(z) + \bar{v}_{12}(z)V_2(z)]\,dz$$

$$\langle M\rangle = -\frac{1}{2}\,\mathrm{Re}\left\{\int\limits_{\Gamma'} [\bar{v}_{11}(z)V_1(z) + \bar{v}_{12}(z)V_2(z)]\,z\,dz\right\} \tag{5.7}$$

On the basis of equations (5.5) and (5.6) and the properties of the functions $U_m(z)$ we can reckon that along the contour Γ_1 there are distributed polar singularities with densities $\bar{v}_{1m}(\zeta)$. Then, setting $N_m = \bar{v}_{1m}(\zeta)\,d\zeta/2\pi i$ in equations (2.4), integrating both parts of them along Γ_1 and making use of formulas (3.12) and (5.6), in place of formulas (2.6) for the functions $\Omega_{1m}(z)$ we can obtain another representation

$$\Omega_{1m}(z) = U_m(z) - \frac{1}{2\pi}\int\limits_0^\infty [(B+1)\overline{H_m(\lambda)}\,e^{-i\lambda z} +$$

$$+ C\overline{H_m(-\lambda)\,e^{i\lambda z}}]\,d\lambda + \frac{1}{2\pi}\int\limits_0^\infty A\,[H_m(\lambda)\,e^{i\lambda z} - H_m(-\lambda)\,e^{-i\lambda z}]\,d\lambda \tag{5.8}$$

Comparing expressions $\bar{v}_{1m}(z) = \Omega_{1m}(z) + F_{1m}(z)$ and (5.5), we obtain

$$V_m(z) = \frac{1}{2\pi}\int\limits_0^\infty \{A\,[H_m(\lambda)\,e^{i\lambda z} - H_m(-\lambda)\,e^{-i\lambda z}] - [(B+1)\overline{H_m(\lambda)}\,e^{-i\lambda z} +$$

$$+ C\overline{H_m(-\lambda)}\,e^{i\lambda z}]\}\,d\lambda + A_m e^{-i\lambda_0 z} + B_m e^{-i\nu z} + C_m e^{i\lambda_0 z} \tag{5.9}$$

Substituting $V_m(z)$ from (5.9) in formulas (5.7), we find that

$$\langle X\rangle = \mathrm{Im}\,\{A_0\alpha_0 + B_0\alpha_1(\lambda_0) + B_\nu\alpha_1(\nu) + C_0\alpha_1(-\lambda_0)\} \tag{5.10}$$

$$\langle Y\rangle = \frac{1}{4\pi}\int\limits_0^\infty [(B+1)\,k_1(\lambda) + Ck_1(-\lambda)]\,d\lambda + A_0\,\mathrm{Re}\,[\alpha_0] \tag{5.11}$$

$$\langle M\rangle = -\frac{1}{2}\operatorname{Im}\left\{\frac{1}{2\pi}\int_0^\infty\left[A\frac{dk_2(\lambda)}{d\lambda} + (B+1)k_3(\lambda) + Ck_3(-\lambda)\right]d\lambda + \right.$$
$$\left. + A_0[\alpha_2(\lambda_0) - \alpha_2(-\lambda_0)] + B_0\alpha_3(\lambda_0) + B_\nu\alpha_3(\nu) + C_0\alpha_3(-\lambda_0)\right\} \qquad (5.12)$$

where

$$k_1(\lambda) = |H_1(\lambda)|^2 + |H_2(\lambda)|^2, \qquad k_2(\lambda) = H_1(\lambda)H_1(-\lambda) + H_2(\lambda)H_2(-\lambda)$$
$$k_3(\lambda) = \overline{H_1(\lambda)}H_1'(\lambda) + \overline{H_2(\lambda)}H_2'(\lambda), \qquad \alpha_0 = H_1(-\lambda_0)H_2(\lambda_0) - H_1(\lambda_0)H_2(-\lambda_0)$$
$$\alpha_1(\lambda) = \overline{H_1(\lambda)}H_2(\lambda), \qquad \alpha_2(\lambda) = H_1(-\lambda)H_2'(\lambda) - H_2(-\lambda)H_1'(\lambda) \qquad (5.13)$$
$$\alpha_3(\lambda) = \overline{H_1(\lambda)}H_2'(\lambda) - \overline{H_2(\lambda)}H_1'(\lambda)$$

3. In the fundamental formulas of the problem (5.3), (5.10), (5.11) and (5.12) there appear Kochin's functions $H_m(\lambda)$, which can be found in the following way: solving equations (4.12) and substituting the resulting functions $\gamma_m^{\circ}(s)$ and $\gamma(s, \lambda)$ in formulas (4.14), we determine from them the functions $H_m^{\circ}(\lambda)$, $h(\lambda, \Lambda)$ and $H(\lambda, \Lambda)$; substituting the latter in formulas (4.17) and using formulas (4.16), we obtain from formulas (4.15) and (4.2) the coefficients A_m', B_m' and C_m'. Finally, determining the functions $\gamma_m(s)$ from (4.11) and substituting them in formulas (3.1), we find the functions $H_m(\lambda)$.

But for sufficiently great depth of submersion of the body the functions $H_m(\lambda)$ can be found approximately by substituting in formulas (3.12) in place of $\overline{v}_{1m}(z)$ the corresponding functions $\overline{v}_{m\infty}(z)$ for oscillations of a body in infinite liquid.

In conclusion we note that when $\rho_1 = \rho_2$ we obtain Kochin's case [1]; setting everywhere in our formulas $r = 0$, we arrive at the corresponding formulas of Kochin [1]. If now we set $\rho_2 = \infty (r = \infty)$, then we arrive at the formulas of Khaskind [2].

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